Is sound field control determined at all frequencies? How is it related to numerical acoustics?

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ABSTRACT

Long before audio technologists have been researching the acoustic reproduction of entire sound fields using loudspeaker arrays, numerical acousticians began to study the solution of boundary integral equations. This is particularly interesting because the fundamental question of uniqueness also appeared in numerical acoustics much earlier than it recently did in the theory of sound field synthesis, where it still appears to be pending. There were two main approaches that proved non-uniqueness at certain frequencies to be soluble by enforcing all the necessary mathematical constraints. Both are directly applicable to generally ensure uniqueness in the sound field synthesis theory based on free field monopole sources.

1. INTRODUCTION

Most sound field synthesis approaches aim at controlling a sound field enclosed by a surrounding loudspeaker arrangement. The control should ideally be able to create any interior sound field.

Recently, the theory of sound field control has been established more clearly than in the past, e.g., by the works of Ise, Kimura, Fazi, Spors, [1, 2, 3, 4], and it appears that the basic principle largely works under general circumstances, applying to several sound field reproduction techniques (Wave Field Synthesis, Boundary Surface Control, Higher-Order Ambisonics, and in some sense also Vector-Base Amplitude Panning). However, some of the works raised a fundamental question: *Is sound field control undetermined at some frequencies and modes, for which the air volume enclosed by the surrounding loudspeakers exhibits resonances, [5, 6]?* The problem is frequently referred to as the non-uniqueness problem of sound field synthesis. This contribution reviews the mathematical theory to gain more insight into whether these frequencies really are a fundamental problem to sound field control. To gain mathematical clarification, the basics of the theory are revisited.

Theory commonly simplifies the loudspeaker arrangement to an arrangement of monopole sources in a free sound field, i.e., free field Green's functions. Another abstraction is helpful: The surrounding arrangement of loudspeakers is assumed to be infinitely dense, i.e. a continuous distribution of infinitely many sources on the hull enclosing the sound field. Unlike the theory suggests, loudspeakers of a real loudspeaker arrangement would neither be omnidirectional, nor would they exhibit a perfectly flat frequency response in all directions; neither would they be located in a sound field that is easy to describe, nor would they let pass through any sound from other loudspeakers without diffraction. And normally their number is limited. Despite unrealistic and overoptimistic elements of the theory are easily observed, we need better arguments to clarify whether sound field reproduction is affected by non-uniqueness. In fact, the theory still seems to be surprisingly effective when it comes to perception [7], and it is able to describe most sound field reproduction techniques we apply nowadays. *Is there a general problem of non-uniqueness and missing frequencies in sound field reproduction?*

The mathematical formulation of the infinitely dense source distribution is a Fredholm integral operator of zero index or, in physically terms, the expression single-layer potential formulation is often found [8, 3, 5]. The model using the surrounding hull of monopole sources can be regarded as a simplification of the more general

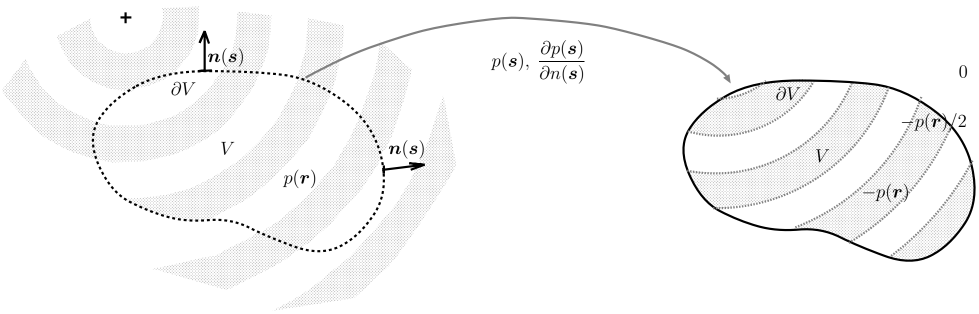

Fig. 1: When taking the values $p(\boldsymbol{s})$ and $\frac{\partial p(\boldsymbol{s})}{\partial n(\boldsymbol{s})}$ on a boundary ∂V around a homogeneous field, the Kirchhoff-Helmholtz integral represents $-p(\boldsymbol{r})$ inside V, $p(\boldsymbol{r})/2$ on ∂V, and 0 outside.

Kirchhoff-Helmholtz integral equation, which additionally involves dipole sources pointing outwards of the hull. It appears that the suspected uniqueness problem is only due to that very simplification step of removing dipoles.

Still, the simplification obtained by discarding the dipole sources of the Kirchhoff-Helmholtz integral equation (KHI) is essential for practical applicability of sound field synthesis as dipole loudspeakers are still impractical. What is more, the simplification has been utilized much earlier to approach boundary integral equations in theoretical acoustics which led to the formulation of the boundary element method (BEM) in numerical acoustics. Not surprisingly, the other disciplines faced similar doubt about uniqueness, but they could clear it up. A matter of fact that we want to involve in our argumentation.

To this end, we are going to explain the historical development of the KHI and its application to acoustic scattering problems. Such problems are of particular interest, because there is a scattering problem that is equivalent to sound field synthesis, as clearly addressed in the work of Fazi [4, 5]. We give a brief historic overview of non-uniqueness in acoustic equations, beginning with Sommerfeld's radiation condition that brings uniqueness to the exterior problem, and finally turning to boundary integral equations of scattering problems that suffer from own uniqueness problems due to resonances of the interior field. Especially, the equivalent scattering formulation discards the dipole sources in the KHI and hereby might introduce resonances of a sound-soft boundary condition.

By addressing all properties, not only the boundary values of the compound boundary integral formulation, we can show that sound field synthesis is generally possible with theories based on surrounding hulls of monopole sources; a model that is directly or indirectly used by nearly all existing sound field reproduction techniques.

2. GREEN, HELMHOLTZ, KIRCHHOFF

Inspired by the new analytical methods introduced in Green's essay in 1828 [9] for the Poisson/Laplace equation, the researchers Hermann von Helmholtz (1860) and Gustav Robert Kirchhoff (1883) reformulated Green's third integral theorem for the wave equation. Hereby, they discovered the equation of Huygens' principle [10], which had been unclear for more than 100 years [11, 12].

2.1. Green's third integral theorem

Green's third integral theorem defines an analytic method applicable to simplify integral relations in scalar potential fields. It is related to Gauß' divergence theorem that equates the flux of a vector field $\boldsymbol{u}(\boldsymbol{r})$, obtained by integration over a closed surface ∂V, with the vector surface element $\mathrm{d}\boldsymbol{S}(\boldsymbol{s})$ pointing outwards, and its divergence $\boldsymbol{\nabla}^{\mathrm{T}}\boldsymbol{u}(\boldsymbol{s})$ integrated over the volume V,

$$\oint_{\boldsymbol{s}\in\partial V} \boldsymbol{u}(\boldsymbol{s})^{\mathrm{T}}\,\mathrm{d}\boldsymbol{S}(\boldsymbol{s}) = \int_{\boldsymbol{s}\in V} \boldsymbol{\nabla}^{\mathrm{T}}\boldsymbol{u}(\boldsymbol{s})\,\mathrm{d}V(\boldsymbol{s}),$$

where del $\boldsymbol{\nabla}^{\mathrm{T}} = [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}]$ is a vector of Cartesian derivatives. Green's theorem uses two functions p and G to construct $\boldsymbol{u}$

$$\boldsymbol{u} = p\,\boldsymbol{\nabla} G - G\,\boldsymbol{\nabla} p,$$

which yields his third integral theorem

$$\oint_{\partial V} \left[p\,\boldsymbol{\nabla} G - G\,\boldsymbol{\nabla} p\right]^{\mathrm{T}} \mathrm{d}\boldsymbol{S} = \int_{V} \left[p \bigtriangleup G - G \bigtriangleup p\right] \mathrm{d}V,$$

where $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator. The functions p and G satisfy the Laplace/Poisson equation. In particular, p satisfies $\triangle p = 0$ inside V and G, the *Green's function*, is a solution whose excitation lies in an exceedingly small point $\triangle G = -\delta$, using δ, the Dirac delta distribution[1], symbolizing the infinitesimal normalized volume excitation

$$\delta(\boldsymbol{r}-\boldsymbol{s}) = \lim_{\varepsilon\to 0} \begin{cases} 0, & \text{for } \|\boldsymbol{r}-\boldsymbol{s}\| > \varepsilon, \\ \frac{3}{4\pi\varepsilon^3}. & \text{for } \|\boldsymbol{r}-\boldsymbol{s}\| \le \varepsilon. \end{cases}$$

Inserting $\triangle p = 0$ and $\triangle G = -\delta$ into the volume integral yields $-\int_V \delta(\boldsymbol{r}-\boldsymbol{s})\, p(\boldsymbol{s})\, \mathrm{d}V(\boldsymbol{s})$. As the integration range is limited, this becomes $-p(\boldsymbol{r})$ for $\boldsymbol{r}\in V$ and 0 otherwise.

2.2. Kirchhoff-Helmholtz Integral (KHI)

Helmholtz [13] and Kirchhoff [14] applied the same procedure to the wave equation. In the frequency domain, p and G satisfy the Helmholtz equation; p satisfies the homogeneous equation $(\triangle + k^2)p = 0$ inside V, and Green's function satisfies $(\triangle + k^2)G = -\delta(\boldsymbol{r}-\boldsymbol{s})$, with the solution $G = \frac{e^{-\mathrm{i}kr}}{4\pi r}$. The integral becomes[2], see Fig. 1,

$$I = \oint_{\boldsymbol{s}\in\partial V} \left[p(\boldsymbol{s})\, \frac{\partial G(\boldsymbol{r}-\boldsymbol{s})}{\partial n(\boldsymbol{s})} - G(\boldsymbol{r}-\boldsymbol{s})\, \frac{\partial p(\boldsymbol{s})}{\partial n(\boldsymbol{s})} \right] \mathrm{d}S(\boldsymbol{s}), \tag{1}$$

and it consists of the scalar fields, p, G, and their outward normal derivatives $\frac{\partial p}{\partial n}$, $\frac{\partial G}{\partial n}$ on every point on ∂V. The value of the integral becomes

$$I(\boldsymbol{r}) = \begin{cases} -p(\boldsymbol{r}), & \boldsymbol{r}\in V, \\ -p(\boldsymbol{r})/2, & \boldsymbol{r}\in\partial V, \\ 0, & \boldsymbol{r}\notin V, \end{cases} \tag{2}$$

which is obtained upon insertion of $\triangle p = -k^2 p$ and $\triangle G = -k^2 G - \delta$ into the volume integral

$$\int_V \left[p \triangle G - G \triangle p \right] \mathrm{d}V(\boldsymbol{s}) = -\int_V \delta(\boldsymbol{r}-\boldsymbol{s})\, p(\boldsymbol{s})\, \mathrm{d}V(\boldsymbol{s}).$$

For the second value[3] $-p/2$ only the part of the exceedingly small δ lying inside V is considered whenever the observation point lies on the surface $\boldsymbol{r}\in\partial V$.

[1] $\boldsymbol{\nabla}$ was not used as a symbol in the original work, δ symbolized the Laplace operator, and the delta distribution differed by a factor $\frac{1}{4\pi}$.

[2] Helmholtz' work uses $\bigtriangledown$ for Laplace and Kirchhoff's $\triangle$, $\boldsymbol{\nabla}$ was not used and the delta distribution was as in Green's work.

[3] This value is unused in the original works of Green, Helmholtz, and Kirchhoff. It appears in the 1960s in boundary integral equations.

3. SIMPLE SOURCE SOUND FIELD CONTROL

According to the values of the integral in Eq. (2), the KHI seems to perfectly resynthesize a sound field p inside V based on its boundary values p and $\frac{\partial p}{\partial n(\boldsymbol{s})}$ on ∂V, but with opposite sign. In practice, the KHI could be simulated by suitable loudspeakers surrounding the audience inside V, with characteristics resembling $G(\boldsymbol{r},\boldsymbol{s})$ and $\frac{\partial G(\boldsymbol{r},\boldsymbol{s})}{\partial n(\boldsymbol{s})}$, and which are driven by the corresponding signals p and $\frac{\partial p}{\partial n(\boldsymbol{s})}$. However, this would still be impractical today as representing the derivative source type requires loudspeakers with dipole directivity[4].

A more practical aim of current sound field synthesis theory is to achieve reconstruction $p_{\mathrm{rec}}(\boldsymbol{r}) = p_{\mathrm{in}}(\boldsymbol{r})$ of an incoming field p_{in} inside V by using only monopole loudspeakers representing $G(\boldsymbol{r},\boldsymbol{s})$. Its general model aims at defining a function $\mu(\boldsymbol{s})$ that suitably drives a hull of surrounding Green's functions, a so-called *simple-source approach* or *single layer potential formulation*,

$$p_{\mathrm{rec}}(\boldsymbol{r}) = \oint_{\partial V} \mu(\boldsymbol{s})\, G(\boldsymbol{r}-\boldsymbol{s})\, \mathrm{d}S(\boldsymbol{s}). \tag{3}$$

Such a formulation could be obtained by making the KHI's dipole expression vanish in Eq. (1). This is obviously achieved by zeroing its coefficient $p(\boldsymbol{s}) = 0$, what imposes a sound-soft boundary condition on ∂V and creates an unknown scattered field p_{sc} outside V due to the sound pressure p_{in} of the impinging waves, hence $p = p_{\mathrm{sc}} + p_{\mathrm{in}}$.

Actually, sound field synthesis intends to do anything else but prescribing rigid or sound-soft boundary conditions as naturally real physical conditions would create resonances. Nevertheless, this step is useful and effective in order to obtain a single layer potential formulation.

3.1. Equivalent sound-soft scattering problem

To describe equivalent scattering, we assume a primary field p_{in} impinging on the volume of a scatterer V_{sc}, see Fig. 2.

[4] Nevertheless, theories exist of how to obtain sound field synthesis with higher-order loudspeaker systems, see [15], and convincing practical applications of higher-order loudspeaker systems are already in use, cf. [16]. Still, today only zero-order loudspeaker systems can meet the quality expectations we have concerning wide-band audio playback of eight octaves. By contrast today, successful directivity synthesis utilizes the poor radiation efficiency higher-order components have at low frequencies. Hereby a size-related upper freuqency bound is introduced, and order-dependent lower frequency bounds due to dynamical restrictions of the transducer excursion.

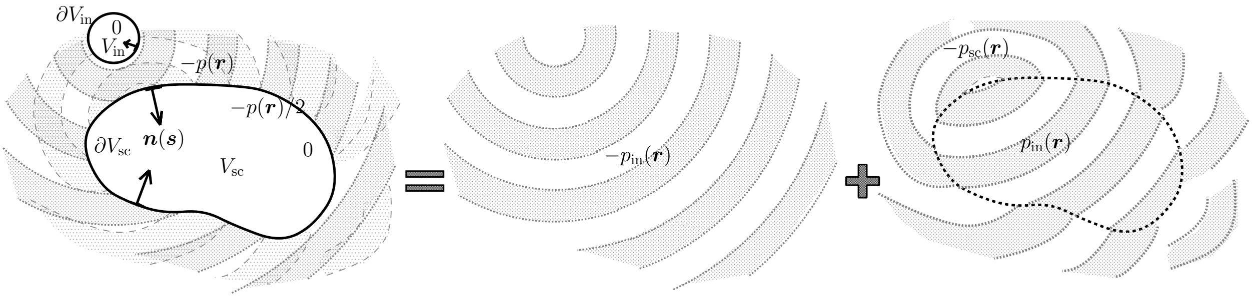

Fig. 2: Kirchhoff-Helmholtz integral applied to define sound-field-synthesis-equivalent sound-soft scattering on ∂V_{sc}. Here, interaction due to thinkable boundary conditions of the virtual source ∂V_{in} is neglected. The represented field must equal the original $-p_{\mathrm{in}}(\boldsymbol{r})$ plus scattered one $-p_{\mathrm{sc}}(\boldsymbol{r})$ outside V_{sc} and the perfect re-synthesis $p_{\mathrm{in}}(\boldsymbol{r})$ inside V_{sc}.

For formal clarity, let us assume an integral I_{in} enclosing the volume V_{in} in which the sources of the incident field p_{in} lie. V_{in} shall be a bounded region outside the scatterer.

Moreover, we assume that the scatterer emits a diffracted field p_{sc} in response. The diffracted wave is represented by the integral I_{sc} around the volume of the scatterer V_{sc}.

Let the surface normals of the compound integral $I_{\mathrm{in}}+I_{\mathrm{sc}}$ point inside the volumes V_{in} and V_{sc}, so that the corresponding volume integral describes the homogeneous field between V_{in} and V_{sc}. Assuming an independent I_{in}, there will be no multiple scattering between both boundaries, and assuming evaluation only outside V_{in}, i.e. $I_{\mathrm{in}}=-p_{\mathrm{in}}$, we get

$$-p_{\mathrm{in}}(\boldsymbol{r})+\oint_{V_{\mathrm{sc}}}\left[p(\boldsymbol{s})\,\frac{\partial G(\boldsymbol{r},\boldsymbol{s})}{\partial n(\boldsymbol{s})}-G(\boldsymbol{r},\boldsymbol{s})\,\frac{\partial p(\boldsymbol{s})}{\partial n(\boldsymbol{s})}\right]\mathrm{d}S(\boldsymbol{s}) = \begin{cases} 0, & \boldsymbol{r}\in V_{\mathrm{sc}},\\ -p(\boldsymbol{r})/2, & \boldsymbol{r}\in\partial V_{\mathrm{sc}},\\ -p(\boldsymbol{r}), & \boldsymbol{r}\notin V_{\mathrm{sc}}.\end{cases} \tag{4}$$

We may impose a boundary impedance condition of the scatterer in this equation, i.e. $p(\boldsymbol{s})=\zeta\frac{\partial p(\boldsymbol{s})}{\partial n(\boldsymbol{s})}$. In $p=p_{\mathrm{in}}+p_{\mathrm{sc}}$ this does not change the incident field p_{in}, which we assumed to be independent, but the scattered one p_{sc} emerging due to the non-free boundary condition. To remove dipoles $\frac{\partial G}{\partial n}$ from the equation, we choose a sound soft impedance $\zeta=0$, i.e. $p(\boldsymbol{s})=0$. Insertion of $p(\boldsymbol{s})=0$ and addition of p_{in} yields

$$-\oint_{V_{\mathrm{sc}}}\frac{\partial p(\boldsymbol{s})}{\partial n(\boldsymbol{s})}G(\boldsymbol{r},\boldsymbol{s})\,\mathrm{d}S(\boldsymbol{s})=\begin{cases} p_{\mathrm{in}}(\boldsymbol{r}), & \boldsymbol{r}\in V_{\mathrm{sc}},\\ p_{\mathrm{in}}(\boldsymbol{r}), & \boldsymbol{r}\in\partial V_{\mathrm{sc}},\\ -p_{\mathrm{sc}}(\boldsymbol{r}), & \boldsymbol{r}\notin V_{\mathrm{sc}}.\end{cases}$$

Focusing on the observation on the integration surface $\boldsymbol{r}\in\partial V_{\mathrm{scat}}$, the formulation of *boundary integral equations* were developed in the 1960s [17, 18, 19]. This brought forward powerful numerical solutions known under the name *boundary element method* (BEM) for acoustic radiation and scattering problems, see [20].

For our purpose, it means that we only need to solve for

$$-\oint_{\partial V_{\mathrm{sc}}}\left[\frac{\partial p_{\mathrm{in}}(\boldsymbol{s})}{\partial n(\boldsymbol{s})}+\frac{\partial p_{\mathrm{sc}}(\boldsymbol{s})}{\partial n(\boldsymbol{s})}\right]G(\boldsymbol{r},\boldsymbol{s})\,\mathrm{d}S(\boldsymbol{s})=p_{\mathrm{in}}(\boldsymbol{r}), \tag{5}$$

with both $\boldsymbol{r},\boldsymbol{s}\in\partial V_{\mathrm{sc}}$, to identify $\frac{\partial p_{\mathrm{sc}}(\boldsymbol{s})}{\partial n(\boldsymbol{s})}$ from $p_{\mathrm{in}}(\boldsymbol{s})$ and $\frac{\partial p_{\mathrm{in}}(\boldsymbol{s})}{\partial n(\boldsymbol{s})}$. According to Fazi [5] and what was said above, Eq. (5) determines the optimal driving function

$$\mu(\boldsymbol{s})=-\frac{\partial p_{\mathrm{inc}}(\boldsymbol{s})}{\partial n(\boldsymbol{s})}-\frac{\partial p_{\mathrm{scat}}(\boldsymbol{s})}{\partial n(\boldsymbol{s})}. \tag{6}$$

4. ABOUT UNIQUENESS

Justified doubt has been raised about the way of determining $\frac{\partial p_{\mathrm{sc}}}{\partial n(\boldsymbol{s})}$ in Eq. (6). If there exist non-trivial solutions p_l of the sound-soft boundary condition $p(\boldsymbol{s})=0$

$$\oint_{\partial V}\frac{\partial p_l(\boldsymbol{s})}{\partial n(\boldsymbol{s})}G(\boldsymbol{r},\boldsymbol{s})\,\mathrm{d}S(\boldsymbol{s})=0,\quad \text{for } \boldsymbol{r}\in\partial V_{\mathrm{sc}},$$

then, for $\boldsymbol{r} \in \partial V_{\mathrm{sc}}$, Eq. (5) stays correct also upon addition

$$\oint_{\partial V} \left[\mu(\boldsymbol{s}) + \sum_l \alpha_l \frac{\partial p_l(\boldsymbol{s})}{\partial n(\boldsymbol{s})} \right] G(\boldsymbol{r},\boldsymbol{s})\, \mathrm{d}S(\boldsymbol{s}) = p_{\mathrm{in}}(\boldsymbol{r}), \quad (7)$$

which means the unknown $\frac{\partial p_{\mathrm{sc}}(\boldsymbol{s})}{\partial n(\boldsymbol{s})}$ is not uniquely determined by Eq. (5). It is a poor consolation that such solutions can only exist at specific frequencies.

This section shows how non-uniqueness was mastered in numerical acoustics by recognizing that a single boundary integral equation degenerates at modal frequencies, and by adding additional constraints.

4.1. Discussion of uniqueness in acoustics

Between the 18th and 19th century, uniqueness was not regarded as a major concern in acoustics. Enclosed modal fields were considered as easy to work with, and when describing acoustic fields radiating towards infinity, researchers knew how to tell useless solutions of the wave equation from the useful ones, e.g. [21].

Sommerfeld recognized the arbitrariness of this common approach and considered a radiation condition [22, 23, 24] to bring rigor and uniqueness to radiating fields.

With the boundary integral equations emerging in the 1960s[5], [17, 18, 19], researchers began to notice there being non-uniqueness problems in single-layer or double-layer potential boundary integral equations; the same problem as we are discussing using Eq. (7) but normally with ideally rigid rather than ideally soft boundaries.

The non-uniqueness problem was first gone unnoticed by some researchers, e.g. [19], because it requires to sharply hit the modal frequencies in numerical simulations. Nevertheless, researchers stayed confident that the problem was avoidable [8, 26] and developed solutions.

A good overview of the solutions is given in [25].

4.1.1. Null-field condition: Schenck's CHIEF point method

Regardless of the boundary condition applied to the KHI, it must become zero outside its volume integration range. In the case of the scattering problem, the volume with the non-zero values lies between V_{in} and V_{sc}.

Non-trivial solutions of the homogeneous boundary condition $p(\boldsymbol{s}) = 0$ for ∂V_{sc} will not be zero inside V_{sc}. Therefore, adding null-field constraints of the KHI to the boundary integral equation is sufficient to rule out non-uniqueness, see Schenck [27]. In fact this means for our problem that Eq. (5) not only needs to be fulfilled for $\boldsymbol{r} \in \partial V_{\mathrm{sc}}$ but also for points $\boldsymbol{r} \in V_{\mathrm{sc}}$, the CHIEF points.

[5] The authors could not get earlier work (1930s, Kuprdaze), see [25].

4.1.2. Surface derivative of pressure: Burton-Miller method

In addition to the fact that single boundary integral equations fail to fulfill the null-field condition inside the scatterer at the interior mode frequencies, they also violate a compatibility condition of the surface normal derivative.

Despite adding null conditions inside the scatterer already works, it is not the only way to obtain uniqueness. Alternatively, Burton and Miller [28] showed that requiring a second boundary integral equation to be fulfilled also provides uniqueness. In addition to Eq. (5), we may demand for $\boldsymbol{r} \in \partial V_{\mathrm{sc}}$ after deriving Eq. (4) by $\frac{\partial}{\partial n(\boldsymbol{r})}$

$$-2 \oint_{\partial V} \left[\frac{\partial p_{\mathrm{in}}(\boldsymbol{s})}{\partial n(\boldsymbol{s})} + \frac{\partial p_{\mathrm{sc}}(\boldsymbol{s})}{\partial n(\boldsymbol{s})} \right] \frac{\partial G(\boldsymbol{r},\boldsymbol{s})}{\partial n(\boldsymbol{r})} \, \mathrm{d}S(\boldsymbol{s}) = -\frac{\partial p_{\mathrm{sc}}(\boldsymbol{r})}{\partial n(\boldsymbol{r})} + \frac{\partial p_{\mathrm{in}}(\boldsymbol{r})}{\partial n(\boldsymbol{r})}. \quad (8)$$

5. HIGHER-ORDER AMBISONICS (HOA)

Higher-order Ambisonics [29] considers a spherical playback region V_{sc}. Its surface ∂V_{sc} is one of a centered sphere of the radius R described by $\boldsymbol{s} = \mathrm{R}\,\boldsymbol{\theta}_{\mathrm{s}}$, using the direction vector $\boldsymbol{\theta} = [\cos\varphi \sin\vartheta, \sin\varphi \sin\vartheta, \cos\vartheta]^{\mathrm{T}}$ with its azimuth and zenith angles, φ, and ϑ. The single layer potential formulation Eq. (3) is either determined by solving its inhomogeneous Helmholtz equation $(\triangle + k^2) = -f(\boldsymbol{\theta})\,\delta(r-\mathrm{R})/\mathrm{R}^2$, e.g. [30], or by solving its spherical boundary integral equation Eq. (5), see [5].

The latter case is more insightful here, as it illustrates the above discussion. As G is rotationally symmetric around the direction $\boldsymbol{\theta}_s$, we may re-express Eq. (5) for $r \leq \mathrm{R}$ as

$$-\oint_{\boldsymbol{\theta} \in \mathbb{S}^2} \frac{\partial p(\mathrm{R}\boldsymbol{\theta}_{\mathrm{s}})}{\partial n(\mathrm{R}\boldsymbol{\theta}_{\mathrm{s}})} G(\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}_{\mathrm{s}}, r, \mathrm{R})\, \mathrm{R}^2 \, \mathrm{d}\boldsymbol{\theta}_{\mathrm{s}} = p_{\mathrm{in}}(r\,\boldsymbol{\theta}), \quad (9)$$

which is a spherical convolution integral [31]. In spherical basis solutions, the incident (finite at $r = 0$) and scattered (Sommerfeld condition at $r \to \infty$) fields, as well as the Green's function for $r \leq \mathrm{R}$ are described as

$$p_{\mathrm{in}}(r\,\boldsymbol{\theta}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} b_{nm}\, j_n(kr)\, Y_n^m(\boldsymbol{\theta}), \quad (10)$$

$$p_{\mathrm{sc}}(r\,\boldsymbol{\theta}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} c_{nm}\, h_n(kr)\, Y_n^m(\boldsymbol{\theta}), \quad (11)$$

$$G(\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}_{\mathrm{s}}, r, \mathrm{R}) = -\mathrm{i}\,k \sum_{n=0}^{\infty} \sum_{m=-n}^{n} Y_n^m(\boldsymbol{\theta}) Y_n^m(\boldsymbol{\theta}_{\mathrm{s}})\, j_n(kr)\, h_n(k\mathrm{R}). \quad (12)$$

Here, $j_n(kr)$, $h_n(kr)$ are the spherical Bessel and Hankel functions, $Y_n^m(\boldsymbol{\theta})$ are spherical harmonics, and b_{nm}, c_{nm} are the wave spectra of the incident and radiating fields. Decomposed in n and m, the convolution integral Eq. (9) for $r \leq \mathrm{R}$ becomes, using $\frac{\partial}{\partial n(\mathrm{R}\boldsymbol{\theta}_{\mathrm{s}})} = -k \frac{\partial}{\partial k\mathrm{R}}$,

$$-\mathrm{i}(k\mathrm{R})^2 \left[b_{nm} j_n'(k\mathrm{R}) + c_{nm} h_n'(k\mathrm{R}) \right] j_n(kr) h_n(k\mathrm{R}) = b_{nm} j_n(kr). \quad (13)$$

Non-unique. Evaluating Eq. (13) at $r = \mathrm{R}$, the remaining equation has the structure $A\, j_n(k\mathrm{R}) = B\, j_n(k\mathrm{R})$, and it is therefore undetermined whenever $j_n(k\mathrm{R}) = 0$.

Unique. Otherwise, Eq. (13) can be divided by $j_n(k\mathrm{R})$. After simplifications given in the appendix, we get the scattered wave spectrum c_{nm} from b_{nm}

$$c_{nm} = -b_{nm} \frac{j_n(k\mathrm{R})}{h_n(k\mathrm{R})}. \quad (14)$$

Whenever $j_n(k\mathrm{R}) = 0$, the **CHIEF** method corresponds to solving Eq. (13) at an interior radius $r < \mathrm{R}$ for which $j_n(kr)$ permits division and hereby also yields Eq. (14).

Whenever $j_n(k\mathrm{R}) = 0$, the **Burton-Miller** method corresponds to deriving Eq. (13) by r before setting $r = \mathrm{R}$ so that we can divide by $j_n'(k\mathrm{R})$ instead to obtain Eq. (14),

$$-\mathrm{i}\,(k\mathrm{R})^2 \left[b_{nm} j_n'(k\mathrm{R}) + c_{nm} h_n'(k\mathrm{R}) \right] j_n'(k\mathrm{R}) h_n(k\mathrm{R}) = b_{nm} j_n'(k\mathrm{R}). \quad (15)$$

Note that we derived towards the inside of V_{sc} here. Division by $j_n'(k\mathrm{R})$ is possible whenever $j_n(k\mathrm{R}) = 0$ because there is only one common zero for $j_n'(k\mathrm{R}) = j_n(k\mathrm{R}) = 0$ at $k\mathrm{R} = 0$ and $n > 0$.

The appendix shows how to obtain the driving function μ for synthesis with Eq. (3) from Eq. (14). For synthesizing a virtual point source, we choose $b_{nm} = -\mathrm{i}k\, h_n(kr_0)\, Y_n^m(\boldsymbol{\theta}_0)$ and get

$$\mu(\boldsymbol{s}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{h_n(kr_0)}{R^2 h_n(k\mathrm{R})} Y_n^m(\boldsymbol{\theta}_0) Y_n^m(\boldsymbol{\theta}_{\mathrm{s}}). \quad (16)$$

6. WAVE FIELD SYNTHESIS (WFS)

From the boundary element method, a high-frequency formulation of the KHI is known which is particularly interesting for deriving Wave Field Synthesis (WFS).

The explicit form of the free field Green's function for $r = \|\boldsymbol{r} - \boldsymbol{s}\|$ in three dimensions is

$$G(r) = \frac{e^{-\mathrm{i}kr}}{4\pi r}. \quad (17)$$

Its normal derivative in the KHI is obtained by applying the chain rule $\boldsymbol{n}^{\mathrm{T}} \frac{\partial r}{\partial(\boldsymbol{r}-\boldsymbol{s})} \frac{\partial}{\partial r} G(r)$, see also Eq. (1), Fig. 1,

$$\frac{\partial G(r)}{\partial n(\boldsymbol{s})} = \frac{\boldsymbol{n}^{\mathrm{T}}(\boldsymbol{r}-\boldsymbol{s})}{r} (-1) \frac{1 + \mathrm{i}kr}{r} G(r). \quad (18)$$

Inserting into Eq. (1) and replacing $\frac{\boldsymbol{n}^{\mathrm{T}}(\boldsymbol{r}-\boldsymbol{s})}{r} = \cos\phi$ yields

$$-\oint_{\partial V} \left[p(\boldsymbol{s}) \cos\phi \frac{1 + \mathrm{i}kr}{r} + \frac{\partial p(\boldsymbol{s})}{\partial n(\mathbf{s})} \right] G(r)\, \mathrm{d}S(\boldsymbol{s}) = -p(\boldsymbol{r})$$

for $\boldsymbol{r} \in V$. As a result we have achieved a representation of the KHI using weighted monopole sources only. However, the weight within the brackets not only depends on the integration variable $\boldsymbol{s}$ but also the observation point $\boldsymbol{r}$. We simplify by the following pair of assumptions

(i) $\cos\phi \to -1$, for $\boldsymbol{r} \in V$ and

(ii) $\mathrm{i}kr \gg 1$, to obtain

$$p(\mathbf{r}) = \oint_{\partial V} \left[-\mathrm{i}kp(\boldsymbol{s}) + \frac{\partial p(\boldsymbol{s})}{\partial n(\boldsymbol{s})} \right] G(\mathbf{r} - \mathbf{s})\, \mathrm{d}S(\mathbf{s}). \quad (19)$$

This approximation is known as the *high frequency boundary element method* [32].

For representing a point source, $p = G(r_0)$, with $\cos\phi_0 = \frac{\boldsymbol{n}^{\mathrm{T}}(\boldsymbol{r}-\boldsymbol{s}_0)}{r_0}$ and $r_0 = \boldsymbol{r} - \boldsymbol{s}_0$, WFS becomes

$$p(\mathbf{r}) = -\oint_{\partial V} \left[\mathrm{i}k + \cos\phi_0 \frac{1 + \mathrm{i}kr_0}{r_0} \right] G(r_0)\, G(r)\, \mathrm{d}S(\mathbf{s}),$$

which is usually further approximated assuming $\mathrm{i}kr_0 \gg 1$ and $(1 + \cos\phi_0) \approx 2\max\{\cos\phi_0, 0\}$,

$$p(\mathbf{r}) = -\oint_{\partial V} 2\mathrm{i}k \max\left\{ \frac{\boldsymbol{n}^{\mathrm{T}}(\boldsymbol{r}-\boldsymbol{s}_0)}{\|\boldsymbol{r}-\boldsymbol{s}_0\|}, 0 \right\} G(\boldsymbol{r}, \boldsymbol{s}_0) G(\boldsymbol{r}, \boldsymbol{s})\, \mathrm{d}S(\mathbf{s}).$$

The last approximation yields the driving function

$$\mu(\boldsymbol{s}) = -2\mathrm{i}k \max\left\{\frac{\boldsymbol{n}^{\mathrm{T}}(\boldsymbol{r}-\boldsymbol{s}_0)}{\|\boldsymbol{r}-\boldsymbol{s}_0\|}, 0\right\} G(\boldsymbol{r}, \boldsymbol{s}_0) \qquad (20)$$

for three-dimensional WFS [33]. As a result of the maximum operator, only those parts of the integral are activated that support waves entering V. In BEM this is known as *determining the visible elements* [32] and in WFS as *secondary source selection criterion* [34]. The selection criterion, be it the hard one from Eq. (20) or a softer variant obtained from Eq. (19), avoids undamped, artificial interior resonances and thus non-uniqueness.

With that said, WFS can be regarded as a high frequency BEM. In the context of WFS the approximations are referred to as *stationary phase approximation* [35].

7. CONCLUSION

The far-field approximation approach to formulate WFS, and the theoretical ability of controlling all fundamental spherical basis solution in HOA did not imply a problem of non-uniqueness that we discussed here. However, uniqueness hasn't been explicitly proven either.

Bringing known properties of acoustic boundary integrals to mind when discussing the sound field synthesis theory in general, we could show that there is no fundamental non-uniqueness problem of using a single-layer potential formulation.

Problems only occur unless all properties of the Kirchhoff-Helmholtz integral equation are expressed; a reason why a single boundary integral equation fails in case of interior resonances of the equivalent scattering problem. Remedy is found by additionally imposing either constraints on the normal derivative of the boundary integral equation (Burton-Miller method) or constraints regarding values in the enclosed volume (CHIEF point method).

For this reason, we can now say that wave field synthesis, higher-order Ambisonics, or boundary surface control, can always be formulated as to provide unique solutions. This also includes amplitude panning methods, which can be regarded as sound field synthesis approaches limited to interpolating between the sources of the playback facility.

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10. APPENDIX: UNIQUE SOLUTIONS OF THE HOA BOUNDARY INTEGRAL EQUATION

After setting $r = \mathrm{R}$ in Eq. (13) and dividing by $j_n(k\mathrm{R})$, we insert the Wronskian $\frac{1}{\mathrm{i}(z)^2} = h_n'(z)\,j_n(z) - j_n'(z)h_n(z)$ on its right hand side and cancel equal terms and common factors

$$-\cancel{b_{nm}\,j_n'(k\mathrm{R})h_n(k\mathrm{R})} - c_{nm}\cancel{h_n'(k\mathrm{R})}h_n(k\mathrm{R}) = \frac{b_{nm}}{\mathrm{i}\,(k\mathrm{R})^2} = b_{nm}\cancel{h_n'(k\mathrm{R})}\,j_n(k\mathrm{R}) - \cancel{b_{nm}\,j_n'(k\mathrm{R})h_n(k\mathrm{R})}.$$

This finally yiels the solution of Eq. (14).

11. APPENDIX: HOA DRIVING FUNCTION

Inserted in $\mu = k\frac{\partial p_{\mathrm{sc}}}{\partial \mathrm{R}} + k\frac{\partial p_{\mathrm{in}}}{\partial \mathrm{R}}$, we get

$$\begin{aligned}\mu(\boldsymbol{s}) &= k\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[b_{nm}\,j_n'(k\mathrm{R}) + c_{nm}h_n'(k\mathrm{R})\right]Y_n^m(\boldsymbol{\theta}_{\mathrm{s}})\\ &= k\sum_{n=0}^{\infty}\sum_{m=-n}^{n} b_{nm}\frac{j_n'(k\mathrm{R})h_n(k\mathrm{R}) - h_n'(k\mathrm{R})\,j_n(k\mathrm{R})}{h_n(k\mathrm{R})}Y_n^m(\boldsymbol{\theta}_{\mathrm{s}})\\ &= -\sum_{n=0}^{\infty}\sum_{m=-n}^{n} b_{nm}\frac{k\,h_n(kr_0)}{\mathrm{i}(k\mathrm{R})^2 h_n(k\mathrm{R})}\,Y_n^m(\boldsymbol{\theta}_{\mathrm{s}}),\end{aligned} \tag{21}$$

and hereby the driving function Eq. (16).