# Array modes of compact rigid microphone arrays with unconventional shapes

Fabio Kaiser[1], Franz Zotter[2], Filippo Fazi[3]
[1] Rohde-BeSB, 5020 Salzburg, Austria, Email: kaiser@rohde-besb.at
[2] IEM, 8010 Graz, Austria, Email: zotter@iem.at [3] ISVR, SO17 1BJ Southampton, UK, Email: ff1@isvr.soton.ac.uk

## Introduction

Methods to find an acoustic modal representation of basic vibration patterns of the surface of arbitrary geometries were introduced since the beginning of the 1990's [1, 2]. In general, these approaches are based on calculating the singular value decomposition (SVD) of a radiation operator. In this work we are interested in identifying these vibrational patterns (array modes) for unconventional shapes and use these modes for microphone array signal processing in the same way spherical harmonics are used for spherical microphone arrays. Therefor, a scattering problem is defined and the SVD of the scattering operator yields the array modes and their corresponding modal strength. The goal is to find frequency independent and real-valued array modes. Fig. 1 shows a block-diagram of the signal processing necessary for beamforming or sound field reconstruction. Using frequency independent functions reduces the computational effort of the filtering steps considerably.

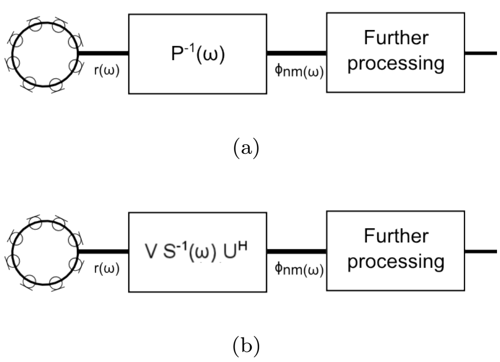



**Figure 1:** Block diagram of the signal processing steps for obtaining the parameters of the acoustic model. (a) shows the generic case where a matrix of transfer functions has to be inverted for each frequency. Fig. (b) shows the simplification using frequency independent array modes..

## Scattering Operator

The scattering problem under investigation can be defined using integral operators [3]. A sketch of the scattering problem is shown in Fig. 2. The sound pressure on the boundary surface $S$ due to a spherical source distribution on $\Omega$ can be written as

$$p^{(tot)}(\boldsymbol{x}) = \int_{\Omega} f(\boldsymbol{y})\, P(\boldsymbol{x}|\boldsymbol{y})\, \mathrm{d}\Omega(\boldsymbol{y})\,, \tag{1}$$

where $f(\boldsymbol{x})$ is the source strenght or a density function and $P(\boldsymbol{x}|\boldsymbol{y})$ is a wave satisfying

$$\begin{cases} (\Delta + k^2)P(\boldsymbol{x}|\boldsymbol{y}) = & -\delta(\boldsymbol{x}-\boldsymbol{y}), \quad \boldsymbol{x} \in \mathbb{R}^3 \backslash V, \boldsymbol{y} \in \Omega, \\ \dfrac{\partial P(\boldsymbol{x}|\boldsymbol{y})}{\partial \mathbf{n}} = & 0\,, \quad \boldsymbol{x} \in S\,, \end{cases} \tag{2}$$

where $S$ is the surface of the scattering body $V$ and $\boldsymbol{x}, \boldsymbol{y}$ are general coordinate vectors. $P$ represents a general wave field, e.g., a Green's function. In operator notation this becomes

$$p^{(tot)}(\boldsymbol{x}) = (\mathcal{P}\, f)(\boldsymbol{x})\,, \tag{3}$$

where the operator $\mathcal{P}$ is defined as acting from the source distribution $\Omega$ to the rigid boundary $S$ ($\mathcal{P} : \Omega \to S$) and represents the integral operation from the right-hand side of Eq. 1.

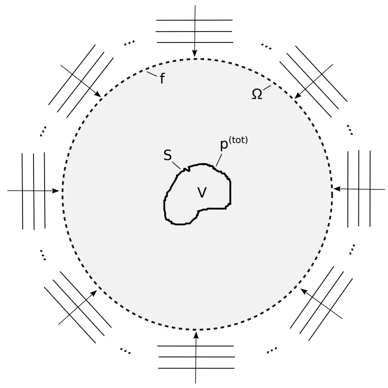



**Figure 2:** Region of definitions for the scattering problem with a continuous distribution of plane waves.

## Spherical Source Distribution

In general, the response of a rigid body $p^{(tot)}$ can be described as a superposition of an incoming sound field $p^{(inc)}$ and a scattered sound field $p^{(scat)}$,

$$p^{(tot)} = p^{(inc)} + p^{(scat)}\,. \tag{4}$$

The scattered sound field can be thought of as a radiating sound field (due to acoustic reciprocity) that forces the normal particle velocity of the total field $p^{(tot)}$ on the boundary surface to zero.

The incoming sound field is generated assuming a spherical distribution of point sources in the far-field. Using an ambisonics approach to describe the incoming sound field [4] a distribution of far-away point sources is given by

$$p^{(inc)}(kr, \boldsymbol{\theta}) = 4\pi \sum_{n=0}^{\infty} \mathrm{i}^n\, j_n(kr) \sum_{m=-n}^{n} \phi_{nm}\, Y_n^m(\boldsymbol{\theta})\,, \tag{5}$$

where $p^{(inc)}$ is the sound pressure with $(r, \boldsymbol{\theta}) \in \mathbb{R}^3$ the spherical coordinates ($\boldsymbol{\theta} = (\varphi, \vartheta)$), $j_n(kr)$ are the spherical Bessel functions and $Y_n^m(\boldsymbol{\theta})$ are the real-valued spherical harmonics [5]. Further, the source strength is given by

$$\begin{aligned} f(\boldsymbol{\theta}) &= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \phi_{nm} Y_n^m(\boldsymbol{\theta}) \,, \\ \phi_{nm} &= \int_{\Omega} f(\boldsymbol{\theta}) Y_n^m(\boldsymbol{\theta}) \, \mathrm{d}\Omega \,. \end{aligned} \quad (6)$$

Using this kind of representation of the incoming sound field the scattering operator can be rewritten to

$$p^{(tot)}(\boldsymbol{x}) = \int_{\Omega} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \phi_{nm} Y_n^m(\boldsymbol{\theta}) \, P(\boldsymbol{x}|\boldsymbol{y}) \, \mathrm{d}\Omega(\boldsymbol{y}) \,, \quad (7)$$

and in compact form

$$p^{(tot)}(\boldsymbol{x}) = \Big(\mathcal{P}_{nm} \, \phi_{nm}\Big)(\boldsymbol{y}) \,, \quad (8)$$

where $\mathcal{P}_{nm}$ is the spherical basis function expanded operator $\mathcal{P}$. The expansion coefficients $\phi_{nm}$ define the sound field of the source and hence they represent the parameters of the acoustic model.

# Scattering Matrix via BEM

The scattered sound field for arbitrary geometries can be written in terms of the Helmholtz integral equation (HIE). Applying a boundary condition for a rigid surface, $\frac{\partial p(\boldsymbol{y})}{\partial \mathbf{n}} = 0$, the total sound pressure is

$$\alpha \, p^{(tot)}(\boldsymbol{x}) = \int_S p^{(tot)}(\boldsymbol{y}) \, \frac{\partial G(\boldsymbol{x}|\boldsymbol{y})}{\partial \boldsymbol{n}} \mathrm{d}S(\boldsymbol{y}) + p^{(inc)} \,, \quad (9)$$

where $\boldsymbol{y} \in S$, $\boldsymbol{x} \in \mathbb{R}^3 \backslash V$ and $\alpha$ is the solid angle (for smooth surface $\alpha = \frac{1}{2}$). Discretizing the boundary of $S$ using $L$ elements and nodes (standard collocation) results in a matrix equation

$$\boldsymbol{C} \boldsymbol{p}^{(tot)} = \boldsymbol{H} \boldsymbol{p}^{(tot)} + \boldsymbol{p}^{(inc)} \,, \quad (10)$$

where $\boldsymbol{H} \in \mathbb{R}^{\mathrm{L} \times \mathrm{L}}$ is the matrix of the normal derivatives of the Green's functions integrated over a single element, $\boldsymbol{p}^{(tot)} \in \mathbb{R}^{\mathrm{L} \times 1}$ and $\boldsymbol{p}^{(inc)} \in \mathbb{R}^{\mathrm{L} \times 1}$ are the vectors of the unknown sound pressures and the sound pressure of the incoming field at the surface nodes in the open field, respectively, and $\boldsymbol{C} \in \mathbb{R}^{\mathrm{L} \times \mathrm{L}}$ is a diagonal matrix containing the solid angles. This discretization scheme is generally know as boundary element method (BEM) [6, 7].

Furthermore, the incoming sound field defined in Eq. 5, written in matrix form is

$$\boldsymbol{p}^{(inc)} = \boldsymbol{R} \boldsymbol{\phi} \,, \quad (11)$$

where $\boldsymbol{\phi} \in \mathbb{R}^{(\mathrm{N}+1)^2 \times 1}$ are the spherical harmonics coefficients and $\boldsymbol{R} \in \mathbb{R}^{\mathrm{L} \times (\mathrm{N}+1)^2}$ is a matrix including the single spherical wave components,

$$\boldsymbol{R} = \begin{pmatrix} j_0(kr_0) Y_0^0(\boldsymbol{\theta}_0) & \cdots & j_{\mathrm{N}}(kr_0) Y_{\mathrm{N}}^{\mathrm{N}}(\boldsymbol{\theta}_0) \\ \vdots & \ddots & \vdots \\ j_0(kr_{\mathrm{L}}) Y_0^0(\boldsymbol{\theta}_{\mathrm{L}}) & \cdots & j_{\mathrm{N}}(kr_{\mathrm{L}}) Y_{\mathrm{N}}^{\mathrm{N}}(\boldsymbol{\theta}_{\mathrm{L}}) \end{pmatrix} , \quad (12)$$

where $N$ is the order of the spherical harmonics expansion and $(N+1)^2$ is the number of spherical waves for which $n \leq N$ . Using Eq. 11 in Eq. 10 the total sound pressure on the boundary surface $S$ is then given by

$$\boldsymbol{p}^{(tot)} = \underbrace{-(\boldsymbol{H} - \boldsymbol{C})^{-1} \boldsymbol{R}}_{\boldsymbol{P}} \boldsymbol{\phi} \,, \quad (13)$$

where $\boldsymbol{P}$ is the scattering matrix and consists of the responses to the single spherical waves. It is the discrete equivalent to the scattering operator Eq. 8 and given by

$$\boldsymbol{P} = (\boldsymbol{p}_{00}, \ldots, \boldsymbol{p}_{nm}, \ldots, \boldsymbol{p}_{NN}) \,, \quad (14)$$

where $\boldsymbol{p}_{nm}$ are the scattered responses due to a single spherical wave and $nm = n^2 + n + m + 1$ is a running index for the spherical harmonics. Note that reconstruction of the sound field or beamforming demand the inversion of the scattering matrix (cf. Figure 1).

## Array Modes

Now, the SVD is applied to the scattering matrix

$$\boldsymbol{P} = \boldsymbol{U} \boldsymbol{\Sigma} \boldsymbol{V}^{\mathbf{H}} , \quad (15)$$

where $\boldsymbol{U} \in \mathbb{R}^{L \times L}$ contains the left singular vectors in its columns, $\boldsymbol{V} \in \mathbb{R}^{(N+1)^2 \times (N+1)^2}$ contains the right singular vectors and $\boldsymbol{\Sigma} \in \mathbb{R}^{L \times (N+1)^2}$ is a diagonal matrix of the singular values in decreasing order

$$|\sigma_1| \geq |\sigma_2| \geq \ldots \geq |\sigma_{\mathrm{N}}| . \quad (16)$$

The left and right singular vectors form a set of orthonormal vectors where $\boldsymbol{U}$ is a matrix of basis vectors which only depend on the geometry of $S$ (the array) and the frequency. These will be called *array modes.* As the incoming sound field is defined by a set of orthonormal functions and if the scattering response does not shape these functions severely the matrix $\boldsymbol{V}$ yields the identity matrix.

As mentioned, the array modes of a sphere should correspond to the spherical harmonics. In analytic terms this is expressed by a plane wave scattered off the rigid sphere with radius $R$

$$p^{(tot)}(kR, \boldsymbol{\theta}) = 4\pi \sum_{n=0}^{\infty} \frac{\mathrm{i}^{n-1}}{(kR)^2 \, h_n^{(2)'}(kR)} \sum_{m=-n}^{n} Y_n^m(\boldsymbol{\theta}) Y_n^{m^*}(\boldsymbol{\theta}_0) \,, \quad (17)$$

where $\boldsymbol{\theta}_0$ represents the incidence direction and $h_n^{(2)'}(kR)$ are the spherical Hankel functions of second order.

# Simulation Results

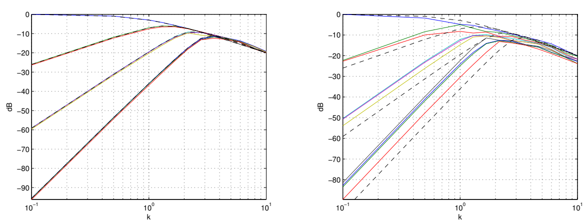
The simulations were done using an axisymmetric BEM formulation [8]. This means that the problem is reduced to one dimension and modal functions can be found that dependent on one coordinate (elevation angle $\vartheta$) and the frequency only. Further, only the generator of the rigid body has to be discretized. The results can be seen as "quasi-continuous", i.e., the sampling density is very high. The incoming sound field is generated for single spherical waves up to third order and positive degree m only. As BEM implementation, the open source Matlab toolbox *OpenBEM* is used [9].

For readability only the results for a sphere (R=1) and one type of cylinder ($\frac{R}{L} = \frac{1}{2}$) are presented here. Fig. 3 shows the computed modal strength (matrix $\mathbf{\Sigma}$ )across frequency for the sphere and the cylinder. The computed array modes (matrix $\boldsymbol{U}$) of three frequencies are shown in Fig. 4 and 5. Figs. 6 and 7 show the computed singular vectors $\boldsymbol{V}$.

## Discussion

For the spherical array (reference) case the method yields array modes and modal strengths corresponding to the spherical wave functions (Eq. 17). In case of the cylinder, a similar behavior of the modes is observable but dependent on the dimensions of the cylinder the array modes are distorted versions of the spherical ones.

In general, in all cases real-valued vectors (or at least constant phase vectors) could be found and the modes are approximately frequency independent up to $kr \approx 1$. Above this frequency the SVD starts to mix the singular functions because the singular values become more uniform. In Fig. 5 the red line corresponding to $kR = 1$ is not at all similar to the mode resulting at $kR = 0.1$ (blue line). Further, the mix of modes can be seen in Fig. 6 and 7. The matrices are not diagonal anymore. It follows that the method is not capable of yielding frequency-independent modes over the whole frequency range of interest.



(a) (b)

**Figure 3:** Generator meshes used for the BEM.

# Conclusions

A method to compute the array modes for a finite cylinder was presented. The modes behave similar compared to the spherical harmonics, but depended on the dimensions of the cylinder, the array modes are distorted versions of the spherical ones. The modes are approximately frequency independent up to $kr \approx 1$.

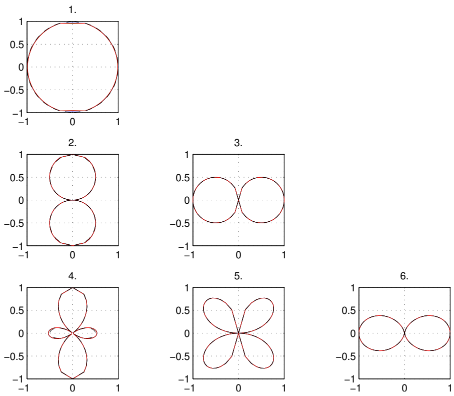


**Figure 4:** Sphere $R = 1$. Singular vectors $\boldsymbol{U}$ at $kR = \{0.1, 0.5, 1\}$ colored and associated Legendre functions in black dashed.

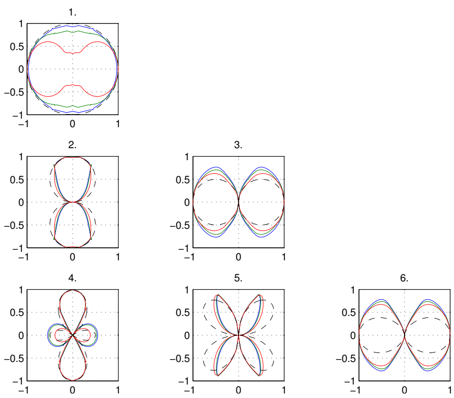


**Figure 5:** Cylinder $\frac{R}{L} = \frac{1}{2}$. Singular vectors $\boldsymbol{U}$ at $kR = \{0.1, 0.5, 1\}$ colored and associated Legendre functions in black dashed.

Above this frequency the SVD starts to mix the singular functions. Hence, the method does not yield frequency-independent array modes.

# References


[1] K. A. Cunefare and M. Noelle Currey, "The radiation efficiency grouping of free-space acoustic radiation modes," *J. Acoust. Soc. Am.*, vol. 109, no. 1, 2001.

[2] P. Nelson and Y. Kahana, "Spherical harmonics, singular-value decomposition and the head realated transfer function," *Journal of Sound and Vibration*, vol. 239, no. 4, pp. 607–637, 2001.

[3] F. Fazi, "Sound field reproduction," Ph.D. dissertation, University of Southampton, 2010.

[4] F. Zotter, H. Pomberger, and M. Frank, "An alternative ambisonics formulation: Modal source strength matching and the effect of spatial aliasing," *AES 126th Convention, Munich, Germany*, 2009.

[5] F. Zotter, "Analysis and synthesis of sound-radiation with spherical arrays," Ph.D. dissertation, University of Music and Performing Arts, Austria, 2009.

[6] R. Ciskowski and C. Brebbia, *Boundary element methods in acoustics.* Computational Mechanics Publications, 1991.

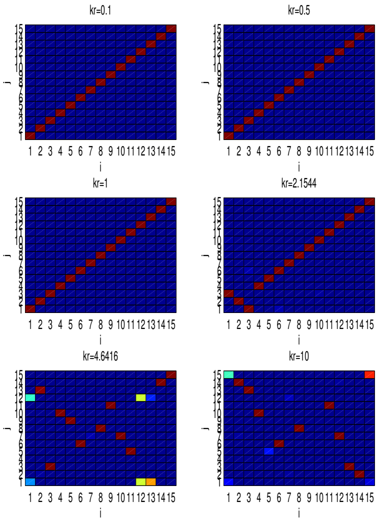



**Figure 6:** Sphere with $R = 1$. Singular vectors $\boldsymbol{V}$ at several frequencies. The color indicates a scale from 0 (blue) to 1 (red) and the indices stand for the i-th column and j-th row of $\boldsymbol{V}$.

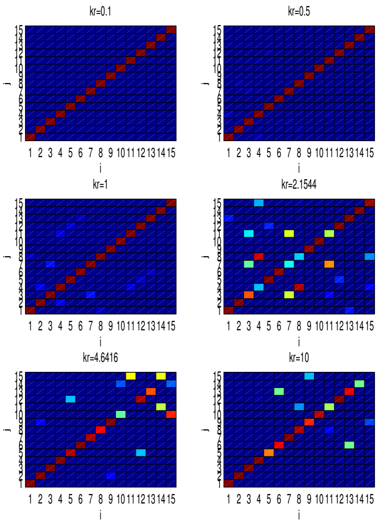



**Figure 7:** Cylinder with $\frac{R}{L} = \frac{1}{2}$. Singular vectors $\boldsymbol{V}$ at several frequencies. The color indicates a scale from 0 (blue) to 1 (red) and the indices stand for the i-th column and j-th row of $\boldsymbol{V}$.

[7] T. Wu, *Boundary Element Acoustics: Fundamentals and Computer Codes.* WIT, 2000.

[8] P. M. Juhl, “The boundary element method for sound field calculations,” Ph.D. dissertation, Thechnical University of Denmark, 1993.

[9] V. Cutanda Henriquez and P. M. Juhl, “Openbem – an open source boundary element method software in acoustics,” *Internoise 2010*, 2010.