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TOWARDS A COMPREHENSIVE ACCOUNT OF VALID AMBISONIC TRANSFORMATIONS

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Abstract: *The classic ambisonic transformations of amplitude scaling, rotation, and mirroring are well known. Their validity can be established by an algebraic analysis of transformation matrices.*
Gerzon and Barton [1] used such a technique to demonstrate the validity of the first-order 'dominance' transformation.
The search for a transformation similar to dominance but applicable to higher-order signal sets is ongoing. Cotterell [2] published a numerical proof that a second-order dominance operation corresponding to Gerzon and Barton's Lorentz transformation is impossible.
Chapman [3] inverted Gerzon and Barton's algebraic method to prove that the only valid first-order transformations are amplitude scaling, rotation, mirroring, and dominance.
In this paper the authors generalise this latter approach and apply it beyond first-order pantophonic matrices, proposing a generalised and extensible approach to the search for ambisonic manipulations.

Key words: transformation matrix, ambisonics.

1 INTRODUCTION

Ambisonics provides a full sphere (or full circle) representation of a soundfield. It is therefore not surprising that the soundfield can be rotated. Such rotations are generally expressed as transformation matrices that can be applied to an ambisonic signal set ($\mathbf{B}$) to produce the required transformed signal set ($\mathbf{B}'$).

It is tempting to presume that rotation is the only possible transformation. However in 1992 Gerzon & Barton [1] demonstrated that 'dominance' (a morphing of the soundfield by a Lorentz transformation) was both possible and of great utility.

We set out here to rigourously explore what valid transformations exist.

By a *valid transformation* in this context we mean a transformation which can be applied to any B–format[3] signal set and which acts on the encoded sound field in such a way as to modify the spatial properties of that sound field —in the case of a simple synthetic sound field consisting only of plane waves, this is equivalent to altering the direction of incidence of (at least some) of those waves. We do not consider either amplitude scaling of the encoded sound field as a whole or filtering operations which act on the timbral characteristics of encoded sounds to be transformations in this sense.

[3] See next section for definition.

2 AMBISONICS

There is no universally agreed notation for ambisonics, and so we set out here the notation used in this paper.

Ambisonics is based on a spherical harmonic description of a soundfield. It is universal that ambisonics uses a set of axes such that x is to the front of the listener, y to the left, and z is upwards. Azimuth is measured from x (0^o), such that due left is at 90^o, etc. Elevation is measured from the horizontal plane, with positive values for directions above the plane. We denote azimuth by θ and elevation by ϕ:

$$0 \leq \theta < 360^o \tag{1}$$

$$-90^o \leq \phi \leq 90^o \tag{2}$$

Any spherical harmonic can be expressed as a function of (θ, ϕ). Alternatively they may be written in terms of the 'direction cosines'.

$$u_x = \cos\phi \cdot \cos\theta \tag{3}$$
$$u_y = \cos\phi \cdot \sin\theta \tag{4}$$
$$u_z = \cos\phi \tag{5}$$

(The direction cosines are the projections onto the coordinate axes of a unit vector pointed in the direction (θ, ϕ).)

The Condon-Shortley phase [4, 6] convention is not used in ambisonics.

Spherical harmonics are described by their degree (l) and their order (m) in that degree. The SH *order* should not be confused with ambisonic order (see below). Each spherical harmonic may then be designated by Y_l^m where l and m are integers, and:

$$l \geq 0 \tag{6}$$

$$-l \leq m \leq l \tag{7}$$

A particular spherical harmonic may be denoted Y_l^m. For a given degree l, there exist $2l+1$ spherical harmonics; the total number of spherical harmonics of all degrees up to some particular l is $(l+1)^2$. The *order* of an ambisonic signal set is equal to the highest *degree* of spherical harmonic needed to describe the signals it contains.

A particular signal within an ambisonic signal set may be denoted B_l^m, paralleling the Y_l^m notation for spherical harmonics. However, it is often convenient to have a single unique integer to identify each signal or spherical harmonic. For this purpose the *ambisonic channel number* (ACN):

$$n = (l+1) \cdot l \; + \; m \tag{8}$$

may be used. We may then simply write

$$\mathbf{B} = (B_0 \; B_1 \; B_2 \; B_3 \ldots) \tag{9}$$

An ambisonic signal set need contain only B_0. It would then be described as being of zero–order. It would in fact be one audio channel of omnidirectional mono. As each further degree is added greater spatial resolution is recorded in the ambisonic signal set.

Spherical harmonics are normalised for most usages. That is there values are adjusted by a constant. Regardless of normalisation the SHs in any given degree have the same values relative to each other. Here (unless stated otherwise) we used full normalisation (or N3D: N for 'normalised', 3D for 'three dimensional'). In consequence the sum of the squares of the SHs within any degree is equal to their number, that is:

$$\sum_{m=-l}^{l} (Y_l^m)^2 = 2l + 1 \tag{10}$$

A comprehensive list of spherical harmonic equations (upto eleventh degree) is published on the Web [5] and Chapman [6] reproduces direction cosine forms upto sixth degree, in these proceedings, as background to a discussion of their symmetry. The higher values in both sets are the work of Philip Cotterell. To ensure no ambiguity in the above definitions, we give a few early values, here, in table 1. (It should be noted that many publications substitute $(1/2)\sin(2\theta)$ for $\sin\theta \cdot \cos\theta$, etc..)

2.1. Two dimensions

It is possible to use circular or cylindrical harmonics to represent a soundfield without any height information.[4] Such signal sets are often described as *pantophonic* (as against *periphonic* for three dimensional sets).

It is also possible to have mixed order sets (where the lower degree(s) are periphonic and the higher degree(s) are pantophonic). These provide greater spatial resolution for sounds in the horizontal plane, and a more limited resolution of 'height'.

As these are both sub-sets of a full periphonic signal set they are of no relevance to the discussion of transformations that follow.

For completeness though, each pantophonic degree contains only two channels, these are B_l^{-l} and B_l^l (the signals related to the sectoral spherical harmonics). So if the entire set is pantophonic the channel count is $2l + 1$.

Classical mixed order sets can be described in *Malham notation*, so that –for example– a fifth order signal set, with the lower three orders in periphony and the upper two pantophonic would have the notation *fffhh* (*f* for 'full' and *h* for 'horizontal').

[4] The nuances of such models do not concern the subject matter of this paper. Whether one needs infinitely long loudspeakers does not effect the mathematics here . . .

Y_0	$=$	1	$=$	1
Y_1	$=$	$\sqrt{3} \cdot \cos\phi \cdot \sin\theta$	$=$	$\sqrt{3} \cdot u_y$
Y_2	$=$	$\sqrt{3} \cdot \sin\phi$	$=$	$\sqrt{3} \cdot u_z$
Y_3	$=$	$\sqrt{3} \cdot \cos\phi \cdot \cos\theta$	$=$	$\sqrt{3} \cdot u_x$
Y_4	$=$	$\sqrt{15} \cdot \cos^2\phi \cdot \sin\theta \cdot \cos\theta$	$=$	$\sqrt{15} \cdot u_x \cdot u_y$
Y_5	$=$	$\sqrt{15} \cdot \sin\phi \cdot \cos\phi \cdot \sin\theta$	$=$	$\sqrt{15} \cdot u_y \cdot u_z$
Y_6	$=$	$\frac{\sqrt{5}}{2} \cdot (3sin^2\phi - 1)$	$=$	$\frac{\sqrt{5}}{2} \cdot (3u_z^2 - 1)$
$\vdots$		$\vdots$		$\vdots$

Table 1: The early spherical harmonic equations used in ambisonics. In N3D. (Table 2 can be used as a concordance between Y_n and Y_l^m namings.)

FuMa	ACN	(l, m)
W	B_0	B_0^0
X	B_3	B_1^1
Y	B_1	B_1^{-1}
Z	B_2	B_1^0
R	B_6	B_2^0
S	B_7	B_2^1
T	B_5	B_2^{-1}
U	B_8	B_2^2
V	B_4	B_2^{-2}
K	B_{12}	B_3^0
L	B_{13}	B_3^1
M	B_{11}	B_3^{-1}
N	B_{14}	B_3^2
O	B_{10}	B_3^{-2}
P	B_{15}	B_3^3
Q	B_9	B_3^{-3}

Table 2: Concordance between FuMa channel letters and ACNs. The B_l^m notation is also shown. There are no FuMa designations beyond third–degree.

2.2. Furse–Malham ambisonics

Previously ambisonic signal sets have used the FuMa scheme. This gives letter codes to the first sixteen possible channels and applies ‘weightings’ rather than normalisations to them.

We give a concordance for channel names in table 2, the scheme though having elegance lacks extensibility and is not used here. A full description can be found in Malham [7] and a recent very brief history in (section 2 of) a paper to this conference [8].

2.3. Other normalisation schemes

Other normalisation schemes exist for spherical harmonics, (see Daniel [9]) of which the commonest is probably semi–normalisation (in three dimensions SN3D).

For semi-normalisation:

$$\sum_{m=-l}^{l} (Y_l^m)^2 = 1 \tag{11}$$

which with equation 10 gives the relationship:

$$Y_l^{m(SN3D)} = \frac{1}{\sqrt{2l+1}} Y_l^{m(N3D)} \tag{12}$$

The same relationship holds true for signals, so that:

$$B_l^{m(SN3D)} = \frac{1}{\sqrt{2l+1}} B_l^{m(N3D)} \tag{13}$$

All such *normalisations* schemes preserve the relative amplitudes of channels within the *same* degree. Thus for sparse matrices (see below), such as rotation matrices, then normalisation is not relevant.

For a non-sparse matrix, such as Gerzon & Barton’s dominance, then any such matrix is specific to one particular normalisation.

Note that FuMa uses ‘weightings’ not normalisation and thus the above does not apply to such sets (except for zero– and first–degree elements).

3 PLANEWAVES

An ambisonic representation of a planewave is created by having each channel contain a signal that is the product of the relevant SH and the ‘input signal’:

$$B_n = Y_n \cdot S \tag{14}$$

where S is the input signal –a mono audio signal.

Planewave signal sets provide useful minimalistic examples for analysing transformations.

Proposition 1 *Any valid ambisonic transformation must be valid for all possible ambisonic signal sets, and therefore must be valid for an ambisonic signal set that represents a planewave.*

Obviously the converse need not necessarily be true.

4 ORDERS

Proposition 2 *Any valid ambisonic transformation must be capable of being truncated so as to be applicable to the next lowest order.*

Though we know of no way to prove the following, it does seem axiomatic that all valid transformations apply to first–order signal sets, that is:

Proposition 3 *The only valid transformations are transformations that can be performed on first–order signal sets.*

5 APPLICATION TO TRANSFORMATION MATRICES

From the above three axioms then if we investigate the possible transformation matrices for first order signal sets we will have a complete set of possible transformations.

The well known rotation matrix is first cited as an example:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) & 0 \\ 0 & \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{15}$$

is a rotation about an axis of an angle α.

It is a *block diagonal sparse* matrix (see p.21 of [10]), that is a matrix of the form:

$$\left(\begin{array}{c|ccc|ccccc} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & * & * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * & * \end{array}\right) \tag{16}$$

The horizontal and vertical lines in the above are purely for readability (they divide the matrix elements into degrees). We are, here, only interested in first order matrices, but a second order example is given to make the pattern clear.

Each part of such a matrix (as delineated above) is referred to as a 'block'. It is possible to refer to such a block, by a notation such as $\mathbf{T_{l=1}}$ for the first degree block (see [9], page 165).

5.1. Sparse first-order matrices

We can write a generalised sparse first-order transformation matrix as:

$$\begin{pmatrix} B'_0 \\ B'_1 \\ B'_2 \\ B'_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & c \\ 0 & d & e & f \\ 0 & g & h & j \end{pmatrix} \times \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix} \tag{17}$$

Now for a planewave

$$\sum_{m=-l}^{l} (B_l^m)^2 = (2l+1)(B_0^0)^2 \tag{18}$$

which is an ugly way of saying the sum of the squares of the spherical harmonics in any order equals their number (for N3D). It applies to signals in a signal set only for planewaves, as for a planewave the input signal (S) in equation 14 cancel out.

Writing the individual values for equation 17 gives us:

$$B'_0 = B_0 \tag{19}$$
$$B'_1 = a \cdot B_1 + b \cdot B_2 + c \cdot B_3 \tag{20}$$
$$B'_2 = d \cdot B_1 + e \cdot B_2 + f \cdot B_3 \tag{21}$$
$$B'_3 = g \cdot B_1 + h \cdot B_2 + j \cdot B_3 \tag{22}$$

Whilst equation 18 gives[5] us:

$$3(B'_0)^2 = (B'_1)^2 + (B'_2)^2 + (B'_3)^2 \tag{23}$$
$$3B_0^2 = B_1^2 + B_2^2 + B_3^2 \tag{24}$$

Combining equations 19 to 24 and simplifying, gives:
$B_1^2 + B_2^2 + B_3^2 = (a \cdot B_1 + b \cdot B_2 + c \cdot B_3)^2 + (d \cdot B_1 + e \cdot B_2 + f \cdot B_3)^2 + (g \cdot B_1 + h \cdot B_2 + j \cdot B_3)^2$
or:
$B_1^2(1 - a^2 - d^2 - g^2) + B_2^2(1 - b^2 - e^2 - h^2) + B_3^2(1 - c^2 - f^2 - j^2) - 2(B_1B_2(ab + de + gh) + B_1B_3(ac + df + gj) + B_2B_3(bc + ef + hj)) = 0$

Now this must be true when $B_2 = B_3 = 0$, that is

$$a^2 + d^2 + g^2 = 1 \tag{25}$$

and likewise

$$b^2 + e^2 + h^2 = 1 \tag{26}$$
$$c^2 + f^2 + j^2 = 1 \tag{27}$$

[5] For those readers trying to reconcile this with SN3D or first-order FuMa: the equation is the same *except* there are no '3's —these cancel out in the next few lines anyway.

Substituting these back in the original equation, gives $B_1B_2(ab+de+gh)+B_1B_3(ac+df+gj)+B_2B_3(bc+ef+hj)=0$, which must be true if only one of B_1, B_2 or B_3 is 0. So we may write

$$ab + de + gh = 0 \quad (28)$$
$$ac + df + gj = 0 \quad (29)$$
$$bc + ef + hj = 0 \quad (30)$$

Now:

Proposition 4 *Every transformation matrix* $\mathbf{T}$ *must have an inverse* $\mathbf{T^{-1}}$*, more simply stated it must be possible to 'undo' any transformation.*
(With the obvious exception that amplitude scaling to 0 *results in an irrecoverable signal set(!!).)*

The caveat does not apply in this case (as there is no amplitude scaling, that is: $B'_0 = B_0$).

Rewriting equation 17 as

$$\mathbf{B'} = \mathbf{T} \times \mathbf{B} \quad (31)$$

then elementary mathematics gives us:

$$\mathbf{T^{-1}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & d & g \\ 0 & b & e & h \\ 0 & c & f & j \end{pmatrix} \quad (32)$$

Repeating the operations above for $\mathbf{T^{-1}}$ will give us:

$$a^2 + b^2 + c^2 = 1 \quad (33)$$
$$d^2 + e^2 + f^2 = 1 \quad (34)$$
$$g^2 + h^2 + j^2 = 1 \quad (35)$$

and

$$ad + be + cf = 0 \quad (36)$$
$$ag + bh + cj = 0 \quad (37)$$
$$dg + eh + fj = 0 \quad (38)$$

This may be summarised as:

Proposition 5 *For a sparse first order transformation matrix, the sum of the squares of the elements in each row of each block equal the sum of the squares of the elements in each column of each block equal one.*

and

Proposition 6 *For a sparse first order transformation matrix, for any two rows (or columns) the products of the two elements in the same column (or row) when summed equal zero.*

(We offer no proof that this applies to the zero-degree block.)

This fits with common experience (e.g. matrix 15).

Now, the number of variables in equations of the form $a^2 + b^2 = 1$ can be reduced by writing $a = \sin\alpha$ and $b = \cos\alpha$, as $\sin^2\alpha + \cos^2\alpha = 1$ always for any value of α.

For the three dimensional situation, we can write[6]:

$$\begin{aligned} 1 &= a^2 + b^2 + c^2 & (39) \\ &= \sin^2\alpha + \cos^2\alpha((b/\cos\alpha)^2 + (c/\cos\alpha)^2) & (40) \\ &= \sin^2\alpha + \cos^2\alpha((b')^2 + (c')^2) & (41) \\ &= \sin^2\alpha + \cos^2\alpha \cdot \sin^2\beta + \cos^2\alpha \cdot \cos^2\beta & (42) \end{aligned}$$

Now we can start simplifying[7] $\mathbf{T}$:

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\alpha & \cos\alpha\cdot\sin\beta & \cos\alpha\cdot\cos\beta \\ 0 & \cos\alpha\cdot\sin\gamma & e & f \\ 0 & \cos\alpha\cdot\cos\gamma & h & j \end{pmatrix} \quad (43)$$

which reduces the variables from nine to seven.

Simplifying further is an ongoing project. It seems probable that a simplification to four variables is possible. Obviously there must be at least three variables to allow for yaw, pitch and roll (for which a matrix can be generated as the product of the three individual matrices). Perhaps a fourth variable is necessary to accommodate mirroring (which is afterall just a rotation through a higher dimension) ...

5.1.1 Caveat

Though the above is not as complete as we would wish, it can be applied to a pantophonic matrix (or to a periphonic matrix where we fix $B'_2 = B_2$).

It will be seen that matrix 15 is a valid solution. It will also be seen that there are trivial variations on this solution, which result for example in backwards rotation or rotation with mirroring. Even with those variations it does not mean that this is the only meaningful solution.

The simple matrix for reflection through the plane $\theta = \alpha$

[6] It is no coincidence that these algebraic formulæ are identical with those for the spherical harmonics!

[7] The authors would be surprised if this process has never been done before. However they remain surprised that from 1784 to 2008 there were no published 'pitch' and 'roll' matrices (see [6] for the brief history) for third degree and above. They would be grateful to be informed of any prior work.

(or, in two dimensions, a line):

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\cos 2\alpha & 0 & \sin 2\alpha \\ 0 & 0 & 1 & 0 \\ 0 & \sin 2\alpha & 0 & \cos 2\alpha \end{pmatrix} \quad (44)$$

satisfies the above conditions. It can be derived from more simple rotation matrix:

If one yaws the soundfield so that the plane one wishes to reflect/mirror through is the plane of the $x-$ and $z-$axes:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos -\alpha & 0 & \sin -\alpha \\ 0 & 0 & 1 & 0 \\ 0 & -\sin -\alpha & 0 & \cos -\alpha \end{pmatrix} \quad (45)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & 0 & -\sin\alpha \\ 0 & 0 & 1 & 0 \\ 0 & \sin\alpha & 0 & \cos\alpha \end{pmatrix} \quad (46)$$

If one then mirrors through that plane

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & 0 & -\sin\alpha \\ 0 & 0 & 1 & 0 \\ 0 & \sin\alpha & 0 & \cos\alpha \end{pmatrix} \quad (47)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\cos\alpha & 0 & \sin\alpha \\ 0 & 0 & 1 & 0 \\ 0 & \sin\alpha & 0 & \cos\alpha \end{pmatrix} \quad (48)$$

And then yaws the plane of reflection back to where it was:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & 0 & \sin\alpha \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\cos\alpha & 0 & \sin\alpha \\ 0 & 0 & 1 & 0 \\ 0 & \sin\alpha & 0 & \cos\alpha \end{pmatrix} \quad (49)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin^2\alpha - \cos^2\alpha & 0 & 2\sin\alpha\cos\alpha \\ 0 & 0 & 1 & 0 \\ 0 & 2\sin\alpha\cos\alpha & 0 & \cos^2\alpha - \sin^2\alpha \end{pmatrix} \quad (50)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\cos 2\alpha & 0 & \sin 2\alpha \\ 0 & 0 & 1 & 0 \\ 0 & \sin 2\alpha & 0 & \cos 2\alpha \end{pmatrix} \quad (51)$$

5.2. Non-block diagonal sparse matrices

These present significantly more variables. Comments are made below on simplifying and considering only the pantophonic case.

Dominance [1] is well known and is quite probably unique (if one includes the variants of with/without mirroring and 'un-dominance').

Another paper to this conference [6] analyses creating higher order planewave files that have had a dominance effect applied to them (which is possible) but demonstrates the impracticality of creating a dominance-transformed higher order file —even for a planewave, unless the source material is available. Indeed the same problem occurs if one merely tries to recreate $(Su_xu_yu_z)$ from a planewave file. (Put in the most basic terms there is a need to divide one audio signal by another. Easily achieved, but as audio signals tend to pass through 0 hundreds or thousands of times per second the results are unuseable.)

6 DIRECTION COSINES

Transformation matrices for u_x, u_y and u_z.

Taking the above propositions to their logical extreme then all block diagonal sparse transformation matrices can be expressed as simple 3×3 matrices for u_x, u_y and u_z (or generalizing as a $n \times n$ matrix for u_1, u_2 $\ldots u_n$ for a n–dimensional soundfield).

$$\begin{pmatrix} u'_y \\ u'_z \\ u'_x \end{pmatrix} = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \times \begin{pmatrix} u_y \\ u_z \\ u_x \end{pmatrix} \quad (52)$$

See the next section.

7 DETERMINING THE ELEMENTS FOR HIGHER DEGREES

Chapman [6] proposes a method for determining the elements of the blocks of a transformation matrix for situations where the above (first degree block) matrix is known.

If *Proposition 3* is valid and there are no unforeseen problems with Chapman's method, then matrix blocks for all higher degrees can be determined (certainly for block diagonal sparse matrices, but the method may be extensible to other matrices if any valid ones exist).

8 DIMENSIONS OTHER THAN THE THIRD

Our discussion has concentrated on three dimensional transformation matrices, as real soundfields are three dimensional. We will though make a few remarks on other dimensions: Both for completeness, and because pantophony is so popular in ambisonics.

8.1. Zero dimensions

A zero dimensional spherical harmonic decomposition of a soundfield has only one channel. Except for amplitude scaling, no transformation is possible.

8.2. One dimension

Has not been considered.

8.3. Two dimensions

The pantophonic case is both mathematically more interesting and practically more relevant.

Applying the above methods it can be shown that the only sparse matrix solution is that of rotation. That is the truncation of matrix 15, or its variants (including mirroring).

The non-sparse case was considered by Chapman [3] for variations in one of the first degree signals. To analyse the situation where both first degree signals are transformed leads to a similar situation to the analysis of matrix 17. Obviously there are valid solutions as dominance at towards an arbitrary angle is possible. It seems unlikely that solutions other than dominance are possible, but it would be useful to prove this.

8.4. Four and higher dimensions

The above methods can be applied to higher dimensional situations. The same generalities apply. The number of variables become significantly larger .... It does not seem impossible that the solution to these equations has not already been determined.

9 CONCLUSIONS

We see two areas where our studies need extending, so as to enable a "comprehensive account".

Firstly a solution to matrix 17. It seems unlikely (but not impossible) that this does not already exist in the literature. The present authors have not though been able to locate such work.

Secondly, the greater challenge of non-block diagonal sparse matrices need addressing.

REFERENCES

[1] Gerzon & Barton Gerzon, Michael A. & Geoffrey J. Barton. "Ambisonic Decoders for HDTV", Audio Engineering Society Preprint, from the 92nd Convention 1992 March 24-27 Vienna. Available for purchase-download from www.aes.org.

[2] Cotterell, Philip, 2002. "On the Theory of the Second–Order Soundfield Microphone". PhD thesis University of Reading. http://www.ambisonic.net/links.html

[3] Chapman, Michael, 2008. "New Dimensions for Ambisonics". Audio Engineering Society Preprint, from the 124th Convention 2008 May 17–20 Amsterdam. Available from http://mchapman.com/amb/hyper.

[4] Weisstein, Eric W. "Condon-Shortley Phase." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/Condon-ShortleyPhase.html

[5] http://ambisonics.ch/standards/channels/

[6] Chapman, Michael. 2009. "Symmetries of Spherical Harmonics: applications to ambisonics". Paper to this symposium.

[7] Malham, Dave. "Higher order Ambisonic systems" (at `http://www.york.ac.uk/inst/mustech/3d_audio/higher_order_ambisonics.pdf`) 2003 abstracted from Malham's MPhil thesis "Space in Music — Music in Space" of the same year.

[8] Michael Chapman, Winfried Ritsch, Thomas Musil, Dhannes Zmölnig, Hannes Pomberger, Franz Zotter & Alois Sontacchi. 2009. "A Standard for Interchange of Ambisonic Signal Sets: Including a file standard with metadata". Paper in this conference's proceedings.

[9] Daniel Daniel, Jérôme. "Représentation de champs acoustiques, application à la transmission et à la reproduction de scènes sonores complexes dans un contexte multimédia" Thèse de doctorat de l'Université Paris 6, 31 juillet 2001. (Also at: `http://gyronymo.free.fr/audio3D/downloads/These.pdf`)

[10] Green, Robin. 2003. "Spherical Harmonic Lighting: The Gritty Details" http://www.research.scea.com/gdc2003/spherical-harmonic-lighting.pdf